Numerical Computation of Singular Control Functions in Trajectory Optimization Problems

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This paper considers the problem of maximizing the final velocity for a two-dimensional climb maneuver of a vehicle in atmospheric flight. The ascent of the vehicle is charaterized by a fixed initial position and a given propellant allocation. The aim is to determine the maneuver time, the throttle setting, the angle between thrust and velocity vector, and the angle of attack, such that a prescribed altitude is reached with zero path inclination and maximal final velocity. For this problem, the necessary conditions of optimal-control theory are derived. It turns out that the throttle setting, which enters linearly into the state equations, in general consists of several bang-bang and singular subarcs. The problem is formulated as a boundary-value problem with switching conditions, which is solved numerically by multiple-shooting techniques. Solutions are presented for various values of the final altitude and various amounts of fuel. Special effort is made to give a survey of the structure of the solutions in dependence of these two parameters.

Introduction

IN many optimal-control applications, problems are formulated for which one or more components of the control vector are bounded and enter linearly into the problem. The minimum principle states that in this situation the corresponding optimal-control variable is either of the bang-bang type (this holds if the switching function has only isolated zeros) or there exist subarcs along which the switching function vanishes identically. These subarcs and the corresponding optimal-control variables are called singular (see Refs. 1–4).

In space navigation many classical problems are of the aforementioned type. One pioneering problem is the so-called Goddard problem,[5] which is to maximize the final altitude for the vertical flight of a rocket. Here, the thrust is a control variable, which is bounded and enters linearly into the state equations. It turns out that through dependence on the given propellant allocation and the aerodynamic approximations, the optimal-thrust history contains a singular subarc, i.e., a subarc with variable-thrust burn.[4,6-9]

In this paper a similar trajectory optimization problem is considered. The problem is to maximize the final velocity for a two-dimensional climb maneuver of a vehicle in atmospheric flight. The maneuver is such that vehicle launch is performed in a nearly vertical position. The aim is to achieve a prescribed altitude with a given propellant allocation and with horizontal velocity. This problem has been investigated by Zlatskiy and Kiforenko.[10] Here, the authors showed that the optimal solution of the problem may contain a singular subarc with respect to the thrust variable. The current authors used a single-shooting-type algorithm and a special procedure in order to compose the optimal trajectory in dependence of the values of the costate variables at the initial time. This method, however, does not seem to be suitable for a direct computation of the solution structure due to dependence on the physical boundary data. In Kumar and Kelley,[11] the problem is considered with respect to the necessary conditions for the singular subarc. These are examined in more detail using the numerical method for Ref. 10. In this paper multiple-shooting techniques are applied,[12-15] for the numerical solution of the problem. To this end we use the necessary conditions of optimal-control theory to establish a boundary-value problem with switching conditions for the state and adjoint variables. This boundary-value problem is solved numerically by the multiple-shooting code BOUNDSCO. Solutions are presented for various values of the final altitude and various amounts of fuel. By application of a homotopy method, we are able to give a survey of the solution structures dependent on these physical parameters.

The computations were carried out in double precision on the SIEMENS 7.882 computer at the University of Hamburg by means of FORTRAN programs.

Problem Statement

The motion of the vehicle is described by the usual fourth-order point-mass equations of motion for planar flight over a spherical Earth and by a mass rate equation to account for the consumption of fuel.

The state variables are:
- r = distance of the vehicle to the gravity center of Earth
- ϕ = range angle
- v = velocity
- γ = flight-path angle
- m = mass

The control variables are
- δ = throttle setting
- ϵ = angle between thrust force and velocity vector
- α = angle of attack

In Fig. 1 the characteristic quantities in atmospheric flight are shown. If T denotes the thrust, D the drag, L the lift, and σ the specific fuel consumption, the (scaled) equations of motion are as follows:

$$\dot{r} = v \sin\gamma \tag{1a}$$

$$\dot{\phi} = \frac{v}{r} \cos\gamma \tag{1b}$$

$$\dot{v} = \frac{1}{m}(T\cos\epsilon - D) - \frac{\sin\gamma}{r^2} \tag{1c}$$

$$\dot{\gamma} = \frac{1}{mv}(T\sin\epsilon + L) + \left(\frac{v}{r} - \frac{1}{vr^2}\right)\cos\gamma \tag{1d}$$

$$\dot{m} = -\sigma T \tag{1e}$$

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The scaling of the system [Eqs. (1)] is such that the quantities are dimensionless. If R denotes the radius of Earth, M the mass of Earth, and μ the gravitational constant, the corresponding characteristic units are

$$[t] = \sqrt{\frac{R^3}{M\mu}} \doteq 804.7 \text{ s}$$

$$[r] = R \doteq 6.36\ E + 6\ m$$

$$[v] = \sqrt{\frac{M\mu}{R}} \doteq 7928 \text{ m s}^{-1}$$

$[m] = m_0 =$ initial mass measured in kg

$$[T] = [L] = [D] = m_0 \frac{M\mu}{R^2}$$

In general, the thrust, drag, lift, and specific fuel consumption depend on the altitude and the velocity:

$$T = T_{\max} \cdot \delta, \qquad T_{\max}(r,v) \geq 0 \tag{2a}$$

$$D = \tfrac{1}{2} S_o \rho(r) v^2 C_D(r,v,\alpha) \tag{2b}$$

$$L = \tfrac{1}{2} S_o \rho(r) v^2 C_L(r,v,\alpha) \tag{2c}$$

$$C_D = C_{DO}(r,v) + C_{D1}(r,v,\alpha)\alpha^2 \tag{2d}$$

$$C_L = C_{L1}(r,v)\,\alpha \tag{2e}$$

$$\sigma = \sigma(r,v) \tag{2f}$$

Here, $\rho(r)$ denotes the atmospheric density, and S_o the reference area of the vehicle. The drag and lift coefficients are C_D and C_L, respectively.

In mathematical terms the problem is to find the final time t_f and piecewise continuous control functions $\delta(t)$, $\epsilon(t)$, $\alpha(t)$, $0 \leq t \leq t_f$, such that the functional

$$I[\delta,\epsilon,\alpha] = -v(t_f) \tag{3}$$

is minimized, subject to the state equations (1), the control constraint

$$0 \leq \delta(t) \leq 1 \tag{4}$$

and the boundary conditions

$$r(0) = r_0, \qquad r(t_f) = r_f \tag{5a}$$

$$\phi(0) = 0 \tag{5b}$$

$$v(0) = v_0 \tag{5c}$$

$$\gamma(0) = \gamma_0, \qquad \gamma(t_f) = 0 \tag{5d}$$

$$m(0) = 1, \qquad m(t_f) = m_f \tag{5e}$$

Necessary Conditions

We separate the right-hand side of the state equations with respect to the thrust:

$$\dot r = \dot r_0, \qquad \dot r_0 := v \sin\gamma, \qquad \dot r_1 := 0 \tag{6a}$$

$$\dot v = \dot v_0 + T\dot v_1, \qquad \dot v_0 := -\frac{D}{m} - \frac{\sin\gamma}{r^2}, \qquad \dot v_1 := \frac{\cos\epsilon}{m} \tag{6b}$$

$$\dot\gamma = \dot\gamma_0 + T\dot\gamma_1, \qquad \dot\gamma_0 := \frac{L}{mv} + \left(\frac{v}{r} - \frac{1}{vr^2}\right)\cos\gamma, \qquad \dot\gamma_1 := \frac{\sin\epsilon}{mv} \tag{6c}$$

$$\dot m = T\dot m_1, \qquad \dot m_0 := 0, \qquad \dot m_1 := -\sigma \tag{6d}$$

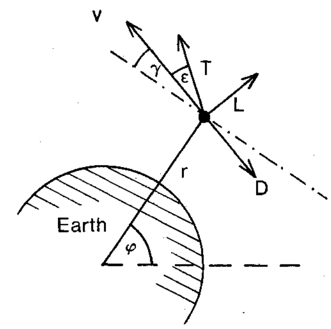

Fig. 1 Characteristic quantities in atmospheric flight.

Let λ_r, λ_v, λ_γ, and λ_m denote the adjoint variables corresponding to the state variables r, v, γ, and m.

Then, the Hamiltonian is given by

$$H = H_0 + TH_1 \tag{7a}$$

$$H_0 = \lambda_r \dot r_0 + \lambda_v \dot v_0 + \lambda_\gamma \dot\gamma_0 \tag{7b}$$

$$H_1 = \lambda_v \dot v_1 + \lambda_\gamma \dot\gamma_1 + \lambda_m \dot m_1 \tag{7c}$$

With the notation from Eqs. (6), we obtain the adjoint differential equations

$$\dot\lambda_r = \dot\lambda_{r0} + T\dot\lambda_{r1} - T_r H_1 \tag{8a}$$

$$\dot\lambda_v = \dot\lambda_{v0} + T\dot\lambda_{v1} - T_v H_1 \tag{8b}$$

$$\dot\lambda_\gamma = \dot\lambda_{\gamma 0} \tag{8c}$$

$$\dot\lambda_m = \dot\lambda_{m0} + T\dot\lambda_{m1} \tag{8d}$$

where

$$\dot\lambda_{r0} = \left(\frac{1}{m} D_r - 2\frac{\sin\gamma}{r^3}\right)\lambda_v + \left[-\frac{1}{mv} L_r + \left(\frac{v}{r^2} - \frac{2}{vr^3}\right)\cos\gamma\right]\lambda_\gamma \tag{9a}$$

$$\dot\lambda_{r1} = \sigma_r \lambda_m \tag{9b}$$

$$\dot\lambda_{v0} = -\sin\gamma\lambda_r + \frac{1}{m} D_v \lambda_v + \left[\frac{1}{mv^2} L - \frac{1}{mv} L_v - \left(\frac{1}{r} + \frac{1}{v^2 r^2}\right)\cos\gamma\right]\lambda_\gamma \tag{9c}$$

$$\dot\lambda_{v1} = \frac{\sin\epsilon}{mv^2}\lambda_\gamma + \sigma_v \lambda_m \tag{9d}$$

$$\dot\lambda_{\gamma 0} = -v\cos\gamma\lambda_r + \frac{\cos\gamma}{r^2}\lambda_v + \left(\frac{v}{r} - \frac{1}{vr^2}\right)\sin\gamma\lambda_\gamma \tag{9e}$$

$$\dot\lambda_{m0} = -\frac{1}{m^2} D\gamma_v + \frac{1}{vm^2} L\lambda_\gamma \tag{9f}$$

$$\dot\lambda_{m1} = \frac{1}{vm^2}\left(v\cos\epsilon\lambda_v + \sin\epsilon\lambda_\gamma\right) \tag{9g}$$

According to Eqs. (3) and (5), the natural boundary conditions are given by

$$\lambda_v(t_f) = -1, \qquad H|_{t=t_f} = 0 \tag{10}$$

The differential equations (1) and (8), together with the boundary conditions [Eqs. (5) and (10)] form the basic boundary-value problem for the numerical solution of the optimal-control problem.

The optimal-control functions are obtained by means of the minimum principle. With respect to the angle of attack, we find the relations

$$0 = C_{L1}\lambda_\gamma - \frac{\partial C_D}{\partial \alpha} v\lambda_v, \qquad 0 \leq -\lambda_v \frac{\partial^2 C_D}{\partial \alpha^2} \tag{11}$$

We assume that the angle of attack is uniquely determined by Eq. (11) as a function $\alpha = \alpha^*(r, v, \lambda_v, \lambda_\gamma)$. This holds, for example, if $C_{D1} > 0$ is independent of α.

With respect to the optimal thrust angle ϵ, the minimum principle yields

$$\sin\epsilon = -\frac{\lambda_\gamma}{\sqrt{\lambda_\gamma^2 + v^2\lambda_v^2}} \tag{12a}$$

$$\cos\epsilon = -\frac{v\lambda_v}{\sqrt{\lambda_\gamma^2 + v^2\lambda_v^2}} \tag{12b}$$

With these relations the switching function $S := mvH_1$ can be rewritten as

$$S := mvH_1 = -\sqrt{\lambda_\gamma^2 + v^2\lambda_v^2} - mv\sigma\lambda_m \tag{13}$$

Thus, with respect to the thrust $T = T_{\max}\delta$, one obtains the bang-bang control

$$\delta = \begin{cases} 0, & \text{if } S > 0 \\ 1, & \text{if } S < 0 \end{cases} \tag{14}$$

Singular Optimal Control

We consider the case of a singular subarc, i.e., we assume that the switching function S vanishes along a subinterval $[\tau_1, \tau_2]$ of $[0, t_f]$. The underlying philosophy for the treatment of singular subarcs is to differentiate the switching function up to a certain order $q = 2p$, such that the total time derivative $s^{(q)}$ contains the control variable explicitly.[2,16] For the problem considered, one finds that $p = 1$. Thus, for the computation of the optimal singular throttle setting $\delta_{\text{sing}}(t)$, $\tau_1 \leq t \leq \tau_2$, one has to determine the first and second total time derivatives of the switching function S.

If one assumes that S vanishes on the interval $[\tau_1, \tau_2]$, one finds, for all t in this subinterval,

$$\begin{aligned} G := & -\sqrt{\lambda_\gamma^2 + v^2\lambda_v^2} \cdot \frac{\mathrm{d}S}{\mathrm{d}t} \\ = & -v\cos\gamma\lambda_r\lambda_\gamma + \frac{v\sin\gamma}{r}\lambda_\gamma^2 - v^2\sin\gamma\lambda_r\lambda_v \\ & -\frac{v^2\cos\gamma}{r}\lambda_v\lambda_\gamma + L\left(\frac{1}{m}\lambda_v\lambda_\gamma - v\sigma^2\lambda_\gamma\lambda_m\right) \\ & + D\left(\frac{1}{mv}\lambda_\gamma^2 + v^2\sigma^2\lambda_v\lambda_m + v^2 m\sigma\sigma_v\lambda_m^2\right) \\ & + \frac{v^2}{m}\lambda_v^2 D_v - \frac{v}{m}\lambda_v\lambda_\gamma L_v \\ & + v^2\sin\gamma m^2\sigma\lambda_m^2\left(-v\sigma_r + \frac{1}{r^2}\sigma_v\right) = 0 \end{aligned} \tag{15}$$

In Eq. (15), one has to substitute the approximations $\sigma = \sigma(r, v)$, $D = D(r, v, \alpha)$, and $L = L(r, v, \alpha)$. Here, σ_r, σ_v, D_v, and L_v denote the partial derivatives of these functions. Finally, one has to substitute $\alpha = \alpha^*(r, v, \lambda_v, \lambda_\gamma)$, where α^* denotes the unique solution of Eq. (11).

To compute the second total time derivative of S, it suffices to compute the partial derivatives of the function $G = G(r, v, \gamma, m, \lambda_r, \lambda_v, \lambda_\gamma, \lambda_m)$ given by the Eq. (15). Notice that G and $\mathrm{d}S/\mathrm{d}t$ vanish identically on the singular subarc.

By a tedious but straightforward calculation, the partial derivatives of G can be determined. These expressions again depend on the approximations σ, D, L, and α^* and the partial derivatives of these functions. The derivatives of σ, D, and L can be determined directly, whereas the derivatives of α^* are obtained by implicit differentiation of the relation (11).

With the assumption that the strengthened Legendre-Clebsch condition holds with respect to the control α, i.e.,

$$\lambda_v \frac{\partial^2 C_D}{\partial \alpha^2} < 0 \tag{16}$$

the resulting relations can be solved for the partial derivatives of α^*.

In order to evaluate $G^{(1)} := \mathrm{d}G/\mathrm{d}t$, one may use the separation of the state and adjoint equations with respect to the thrust. One finds that

$$G^{(1)} = A + TB \tag{17a}$$

$$A := \frac{\partial G}{\partial r}\dot{r}_0 + \ldots + \frac{\partial G}{\partial \lambda_m}\dot{\lambda}_{m0} \tag{17b}$$

$$B := \frac{\partial G}{\partial r}\dot{r}_1 + \ldots + \frac{\partial G}{\partial \lambda_m}\dot{\lambda}_{m1} \tag{17c}$$

In this form, the quantities A and B can be evaluated numerically. It turns out that $B \neq 0$ holds on the singular subarcs, i.e., the singular control is of the order $p = 1$. Thus, the relation $G^{(1)} = 0$ can be solved for the singular control:

$$T_{\text{sing}} = T_{\max} \cdot \delta_{\text{sing}} = -A/B \tag{18}$$

The generalized Legendre-Clebsch condition states that

$$\frac{\partial}{\partial \delta}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial H}{\partial \delta}\right) \leq 0$$

holds along the singular subarc. With Eqs. (15) and (17) and the preceding assumptions, this conditions reduces to

$$B > 0 \tag{19}$$

which has to be satisfied along the singular subarcs.

Numerical Solution

Summarizing the necessary conditions, one obtains the system of first-order differential equations given in Eqs. (1) and (8). The free final time t_f can be considered as an additional variable of the problem and, by the standard transformation

$$t = t_f \cdot \xi, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi} = t_f \cdot \frac{\mathrm{d}}{\mathrm{d}t} \tag{20}$$

the equations are transformed into a system with the independent variable ξ ranging in the interval $0 \leq \xi \leq 1$. Further, the additional trivial differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} t_f = 0 \tag{21}$$

is added to the system. In correspondence to the resulting ten-dimensional system of first-order equations (1), (8), and (21), we have ten boundary conditions given by Eqs. (5) and (10).

Notice, however, that the resulting boundary-value problem is a nonclassical one. One has to substitute the control variables on the right-hand side of the differential equations by the optimal-control laws [Eqs. (11), (12), (14), or (18)], respectively. The switching structure and the switching points themselves are unknown a priori.

For the numerical solution, we estimate the switching structure, i.e., the number and relative position of bang-bang and singular subarcs. Thus, the total number s of switching points τ_k, $k = 1, \ldots, s$, is kept fixed for the numerical iteration procedure, whereas the τ_k themselves become parameters of the boundary-value problem. These parameters are determined such that certain switching conditions are satisfied. If τ_k is a switching point between bang-bang subarcs, i.e., τ_k is a switching point from an $\delta = 0$ subarc to an $\delta = 1$ subarc or vice versa, then the corresponding switching condition is given by

$$S\,|_{t=\tau_k} = 0 \tag{22}$$

Further, if $[\tau_k, \tau_{k+1}]$ denotes a singular subarc, the corresponding switching conditions are given by

$$S\,|_{t=\tau_k} = G\,|_{t=\tau_k} = 0 \tag{23}$$

For the numerical solution of the resulting *boundary-value problem with switching conditions*, we use a multiple-shooting technique.[12,13] The special code BOUNDSCO is an extension of the ordinary multiple-shooting method in the sense that it was developed for the solution of boundary-value problems with switching conditions. For the details of the algorithm, the reader is referred to Refs. 13 and 14.

It should be remarked that the algorithm only guarantees that the switching conditions are satisfied at the solution data for the switching points. It does not ensure that the switching function has the correct sign within the bang-bang subarcs. Therefore, one has to check additional sign conditions after the solution candidate has been computed by the routine. These sign conditions are:

(S1) Test for the correct sign of the switching function within the bang-bang subarcs according to Eq. (14).
(S2) Legendre-Clebsch condition: $\lambda_v \partial^2 C_D / \partial\alpha^2 < 0$.
(S3) Generalized Legendre-Clebsch condition: $B > 0$, within the singular subarcs.
(S4) $0 \leq T_{\text{sing}} \leq T_{\max}$, within the singular subarcs.

Numerical Results

In the following numerical example, we use the rough approximations of the aerodynamic and atmospheric data given in Ref. 10. Here, the characteristic functions $T_{\max}$, C_{DO}, C_{L1}, and σ are approximated by constants. The data are

$$T_{\max} = 3, \qquad \sigma = 2$$

$$S_0\,\rho(r) = be^{-\beta(r-1)}, \qquad b = 6200, \qquad \beta = 500$$

$$C_{DO} = 0.05, \qquad C_{D1} = 0.7, \qquad C_{L1} = 0.7 \tag{24}$$

Notice, however, that the necessary conditions are formulated for much more general approximations as reflected in Eq. (2). The normalized initial data are

$$r_0 = 1.00016, \qquad v_0 = 0.021, \qquad \gamma_0 = 1.48 \tag{25}$$

These data correspond to an initial altitude of 1.0208 km, an initial velocity of 166.5 m/s, and an initial flight-path angle of 84.8 deg.

The numerical solutions are obtained by means of the multiple-shooting code BOUNDSCO. The problem depends on the parameters $h_f := r(t_f) - 1$ (i.e., the final altitude) and m_f (the final mass of the vehicle). We are interested in performing a parametric study of the optimal solutions and the corresponding optimal switching structures that depend on these physical parameters. To this end, we apply a classical continuation or homotopy method.[12,17-19] The basic idea of continuation methods is to construct a homotopy chain of subproblems. In each homotopy step one of the parameter values is changed ($h_f \to h_f + \Delta h_f$ or $m_f \to m_f + \Delta m_f$). The solution for the new parameter set is computed using the solution of the last homotopy step as the initial trajectory for the multiple-shooting method. In this application, the increments (homotopy step sizes) Δh_f or Δm_f are determined heuristically such that Newton's method takes about 7–10 iterations to compute the solution within an accuracy of 10 decimals. For such a typical run, BOUNDSCO takes about 10 s of CPU time on the

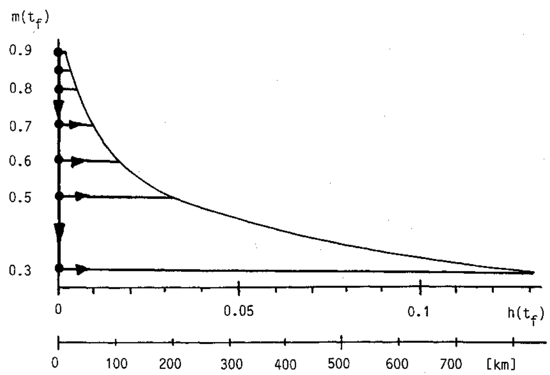

Fig. 2 Homotopy paths with respect to m_f, h_f.

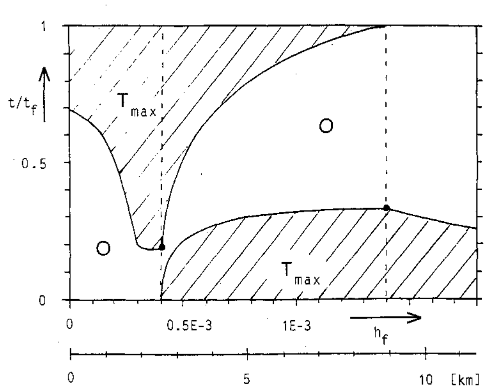

Fig. 3 Optimal switching structure for $m_f = 0.9$.

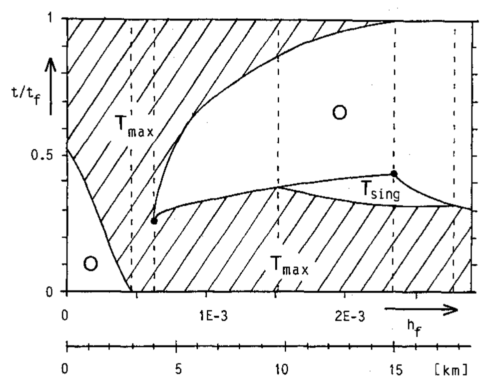

Fig. 4 Optimal switching structure for $m_f = 0.85$.

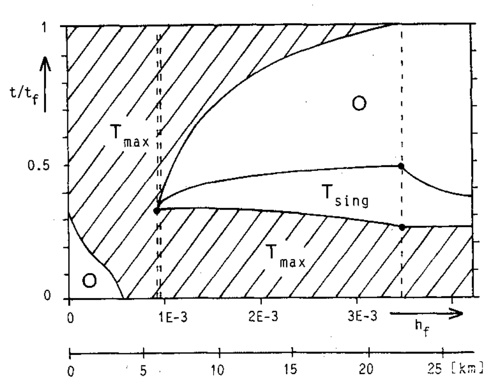

Fig. 5 Optimal switching structure for $m_f = 0.8$.

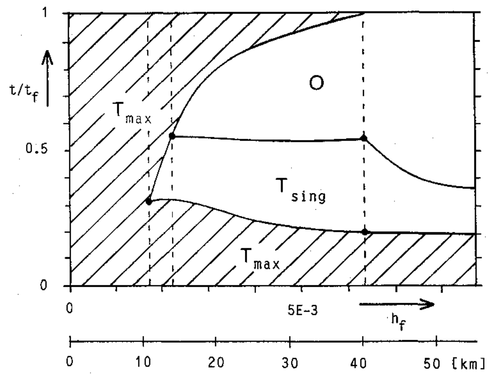

Fig. 6 Optimal switching structure for $m_f = 0.7$.

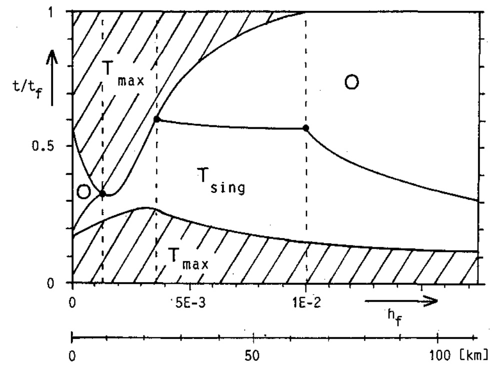

Fig. 7 Optimal switching structure for $m_f = 0.6$.

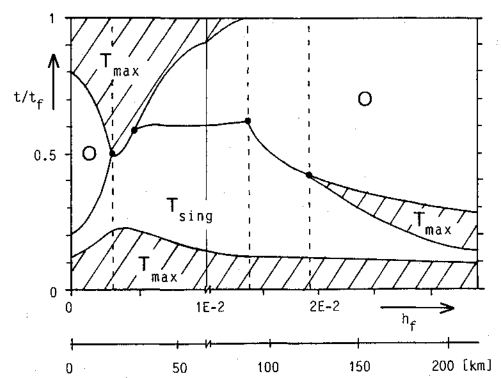

Fig. 8 Optimal switching structure for $m_f = 0.5$.

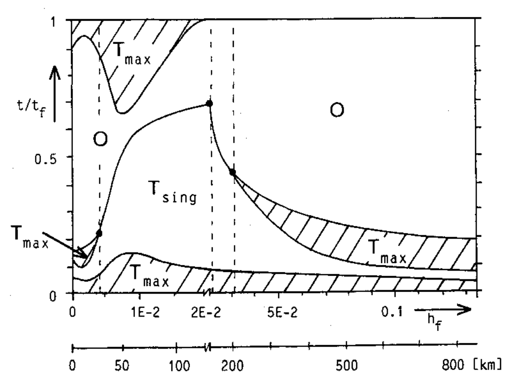

Fig. 9 Optimal switching structure for $m_f = 0.3$.

SIEMENS 7.882 computer. More refined methods for an automatic determination of the homotopy step sizes may be applied (cf., e.g., Deuflhard[20]). In general, however, these methods only work if the switching structure does not change along the continuation path.

The starting point for the homotopy is the optimal solution of the problem for the parameter values

$$m_f = 0.9, \qquad h_f = 1.5\text{E-3} \ (=9.57 \text{ km})$$

For this data set the solution was given in Ref. 10. It consists of two bang-bang subarcs with the switching structure $\delta_{\max} \to 0$. Starting from this solution, homotopy paths of the following type are constructed (see Fig. 2):

(a) m_f fixed, $\quad 0 < h_{f1} \to h_{f2} < h_{f\max}$

(b) $m_{f1} \to m_{f2}$, $\quad h_f = 0$ fixed

The maximal value for the final altitude $h_{f\max}$, which occurs in the homotopy paths (a), depends on the (fixed) final mass. This value represents the maximal altitude that is reachable by the vehicle for a certain prescribed propellant allocation and the restriction that the final velocity be horizontal. In the course of the homotopy (a), the value $h_{f\max}$ is indicated by the condition $v(t_f) \to 0$.

In each homotopy step, the *estimated* switching structure is taken from the last solution trajectory. This structure has to be changed if a solution of the boundary-value problem is computed, which violates the sign conditions ($S1$) or ($S4$). In general, the new switching structure can be identified easily by inspection of the switching function. For instance, the appearance of a singular subarc can be detected by the fact that the derivative $S^{(1)}$ changes sign at a switching point τ between bang-bang subarcs. Thus, at this level of the homotopy parameters, the situation with respect to the switching function at τ corresponds to the situation on a singular subarc (cf. the switching conditions [Eqs. (23)].

In Figs. 3–9, the optimal switching structure is plotted vs the final altitude for increasing values of the propellant allocation. The main results can be stated as follows:

For a low amount of fuel and low final altitude, the optimal control is of the bang-bang type. The switching structure may be: $0 \to \delta_{\max}$, $\delta_{\max}$, $\delta_{\max} \to 0 \to \delta_{\max}$, or $\delta_{\max} \to 0$. Singular subarcs occur for higher final altitudes or larger amounts of fuel. In these cases, up to 50% of the maneuver time is spent on the singular subarc. The structure may be: $\delta_{\max} \to \delta_{\text{sing}} \to \delta_{\max}$, $\delta_{\max} \to \delta_{\text{sing}} \to 0 \to \delta_{\max}$, $\delta_{\max} \to \delta_{\text{sing}} \to 0$, or $\delta_{\max} \to \delta_{\text{sing}} \to \delta_{\max} \to 0$.

Another result is concerned with the optimal-control variables $\epsilon(t)$ (thrust angle) and $\alpha(t)$ (angle of attack). With the assumption of a quadratic polar [cf. Eq. (24)] one obtains

$$\alpha = \frac{C_{L1}}{2C_{D1}} \cdot \tan\epsilon, \qquad \tan\epsilon = \frac{\lambda_\gamma}{v\lambda_v}$$

For the computed optimal trajectories (about 1000 boundary-value problems were solved in the course of the parametric studies shown in Figs. 3–9), the thrust angle is sufficiently small, such that the approximation[21]

$$\epsilon(t) = \frac{2C_{D1}}{C_{L1}} \cdot \alpha(t) \tag{26}$$

is satisfied with an accuracy of 2–3 digits. Thus, as a rule of thumb, Eq. (26) may be used to simplify the optimal-control problem. It should be mentioned, however, that this reduction is only justified in the case of a nearly vertical takeoff.

In Fig. 10, a typical solution trajectory with a singular subarc (homotopy parameters: $m_f = 0.3$, $h_f = 2E\text{-}2$) is shown. The parameter h_f corresponds to a final altitude of 127.6 km. The maximal final velocity is given by $v(t_f) = 0.41886\ 575$, which corresponds to 3320.8 m/s. The whole maneuver takes

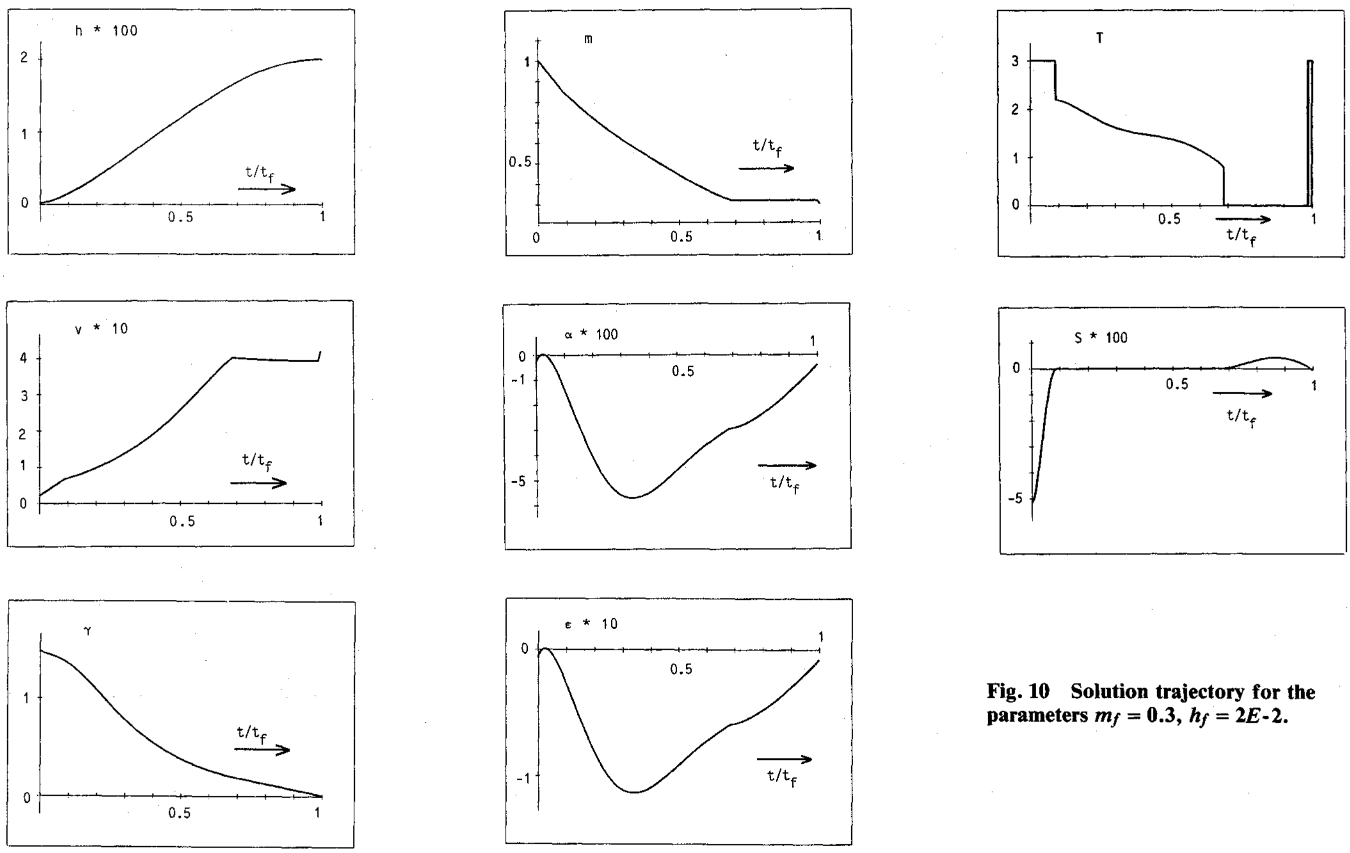

Fig. 10 Solution trajectory for the parameters $m_f = 0.3$, $h_f = 2E$-2.

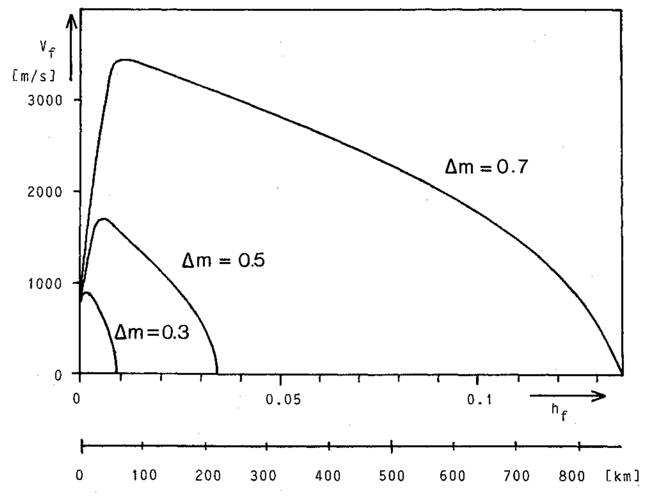

Fig. 11 Maximal final velocity v_f vs the final altitude.

about 234 s. One recognizes that about 135 s of the maneuver time is spent on the singular subarc.

In Fig. 11, the maximized final velocity v_f is plotted vs the prescribed final altitude for different values Δm of the fuel consumption.

Conclusions

This paper deals with the numerical optimizations of climb maneuvers in atmospheric flight. The optimal control functions are computed by solving a boundary-value problem with switching conditions, which is derived from variational calculus. The application of multiple-shooting techiques allows a fast and precise numerical solution of these problems. Further, this method allows the determination of the optimal switching structure with respect to the linear power setting control variable. It is found that, for a low amount of fuel, this is a bang-bang strategy, whereas the optimal-control history contains a singular subarc in the case of larger amounts of fuel.

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